MPDATA meets Black-Scholes
Derivative pricing as a transport problem

Sylwester Arabas & Ahmad Farhat
Chatham Financial Corporation Europe, Cracow, Poland

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MPDATA in a nutshell (Smolarkiewicz 1983, 1984, ...)

transport PDE: $$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(v\psi) = 0$$

$$\psi_i^{n+1} = \psi_i^n - \left[F(\psi_i^n, \psi_{i+1}^n, C_{i+1/2}) - F(\psi_{i-1}^n, \psi_i^n, C_{i-1/2})\right]$$
$$F(\psi_L, \psi_R, C) = \max(C, 0)\cdot\psi_L + \min(C, 0)\cdot\psi_R$$
$$C = v\Delta t/\Delta x$$

upwind

modified eq.: $$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(v\psi) + \underbrace{K\frac{\partial^2\psi}{\partial x^2}}_{\text{numerical diffusion}} + \ldots = 0$$

MEA

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(v\psi) + \frac{\partial}{\partial x}\underbrace{\left[\left(-\frac{K}{\psi}\frac{\partial\psi}{\partial x}\right)\psi\right]}_{\text{antidiffusive flux}} = 0$$

MPDATA: reverse numerical diffusion by integrating the antidiffusive flux using upwind (in a corrective iteration)

$$C'_{i+1/2} = (|C_{i+1/2}| - C^2_{i+1/2})A_{i+1/2}$$
$$A_{i+1/2} = \frac{\psi_{i+1} - \psi_i}{\psi_{i+1} + \psi_i}$$

Black-Scholes ⇝ ("advection-only") transport problem

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} - rf = 0$$

$x = \ln S$

$$\frac{\partial f}{\partial t} + \underbrace{(r - \sigma^2/2)}_{u}\frac{\partial f}{\partial x} + \underbrace{\sigma^2/2}_{-\nu}\frac{\partial^2 f}{\partial x^2} - rf = 0$$

$\psi = e^{-rt}f$

$$\frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} - \nu\frac{\partial^2\psi}{\partial x^2} = 0$$

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}\left[\left(u - \frac{\nu}{\psi}\frac{\partial\psi}{\partial x}\right)\psi\right] = 0$$

Stability condition (for divergent velocity field)

$$C = \left|r - \frac{\sigma^2}{2} + \frac{\sigma^2}{\Delta x}A\right|\frac{\Delta t}{\Delta x} < \frac{1}{2} \quad \rightsquigarrow \quad \lambda^2 = \frac{1}{\sigma^2}\frac{\Delta x^2}{\Delta t} \gtrsim 2$$

(twice more stringent than for the standard first-order-in-time FTCS scheme)

Convergence analysis (second-order in time and space)

- Test case: interest-rate corridor valuation (Fig 1)
- Truncation error estimation (ψ_a: B-S formula):

$$E = \sqrt{\sum_{i=1}^{n_x}\left[\psi_n(x_i) - \psi_a(x_i)\right]^2/(n_x\cdot n_t)}\Bigg|_{t=0}$$

- Time- and space-convergence rates: Fig 2 & Fig 3
- MPDATA (libmpdata++) settings used:
  - one corrective iteration
  - non-oscillatory option (fct)
  - infinite gauge (iga)
  - divergent-flow option (dfl, needed for second-order in time!)

main takeaway: robust explicit alternative to Crank-Nicholson aptly suited for multi-dimensional problems

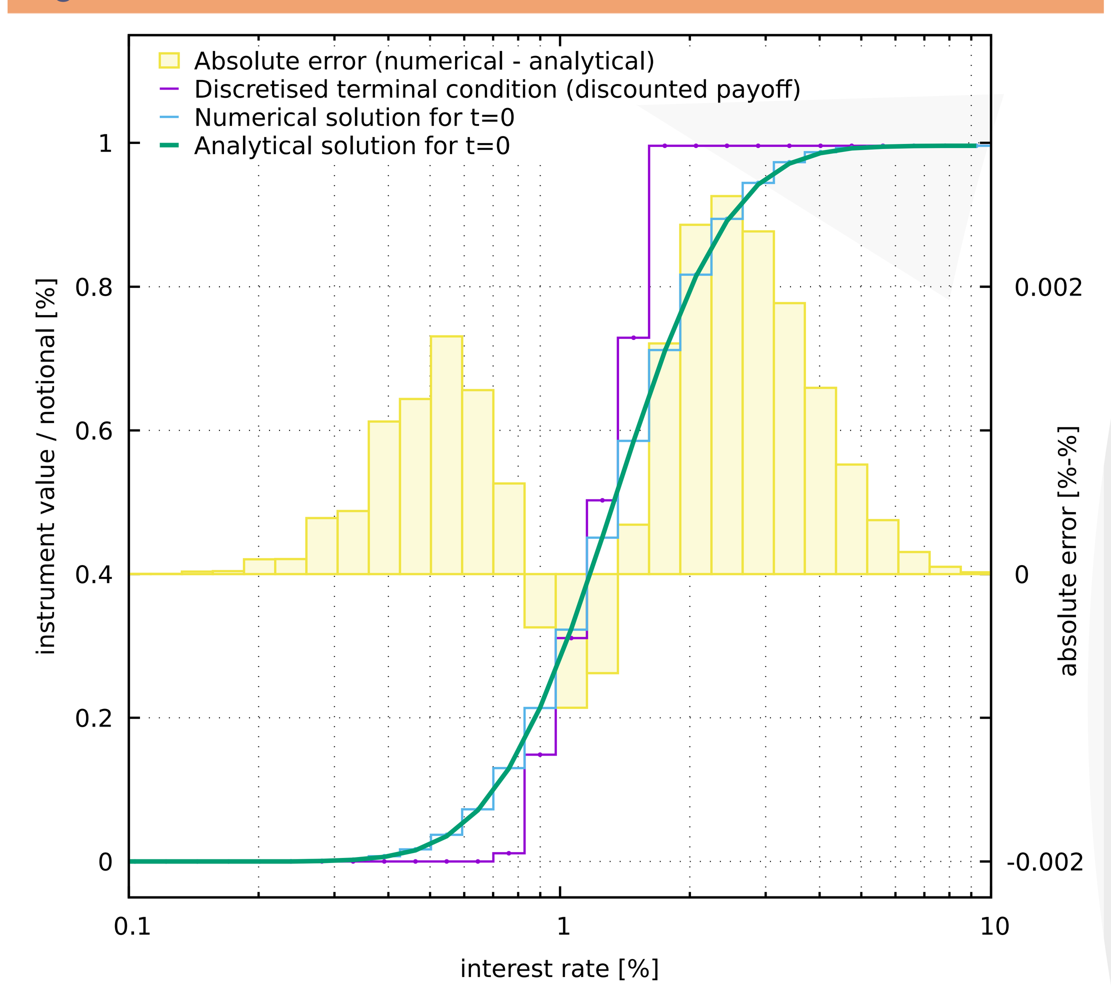
Key characteristics of MPDATA:

- positive definiteness (non-negativity of option price solutions by design, alternative "infinite gauge" formulation for variable-sign ψ)
- monotonicity (no spurious oscillations in the solutions with the "flux corrected transport" option)
- conservativeness and high-order accuracy (second-order in time and space for the basic MPDATA, third-order option available)
- multidimensionality (i.e., antidiffusive fluxes include cross-dimensional terms, as opposed to dimensionally-splitted schemes)

Recent MPDATA developments:

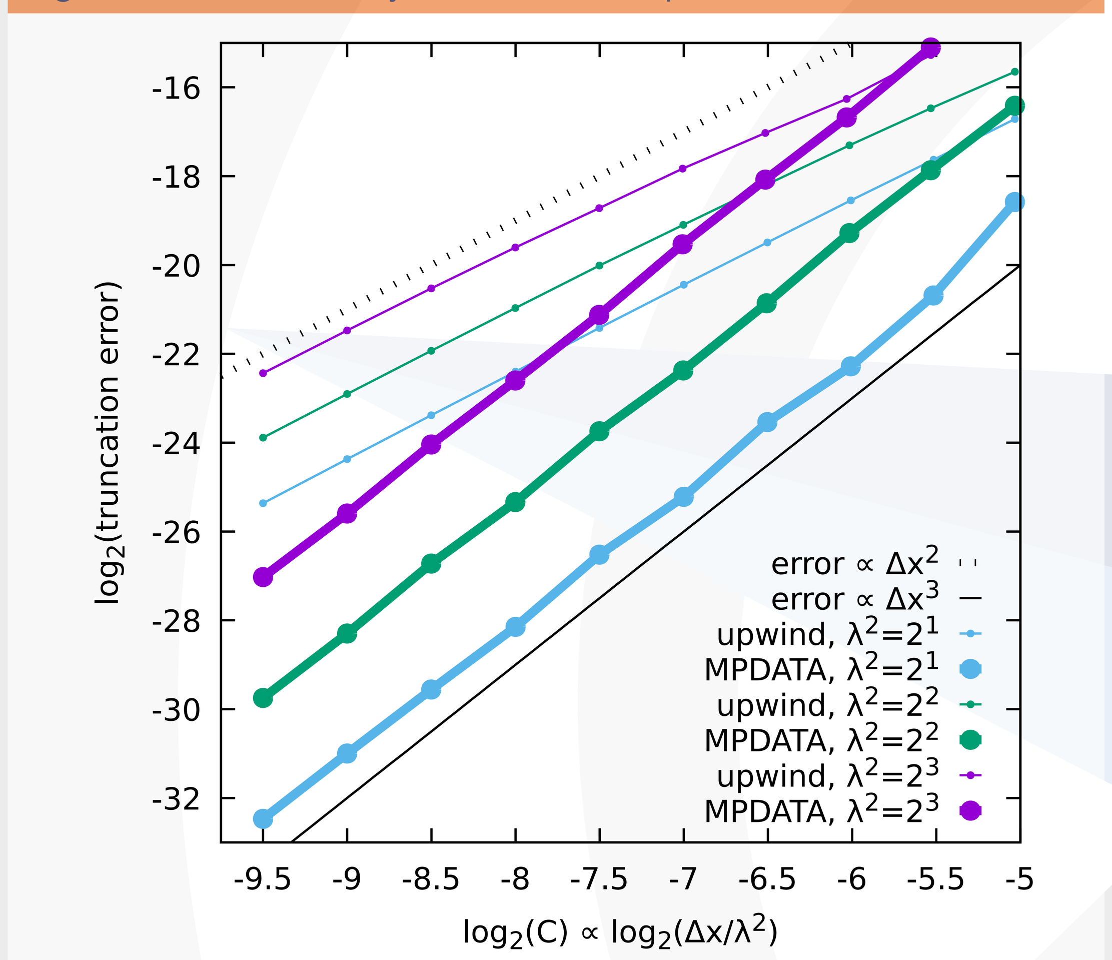
- Unstructured meshes, antidiffusive-flux-controlled mesh adaptivity, and many more (for a review, see Smolarkiewicz, Szmelter, et al., 2016)
- Open-source C++ library: Jaruga, Arabas, et al., 2015

Fig. 1: Interest rate corridor valuation



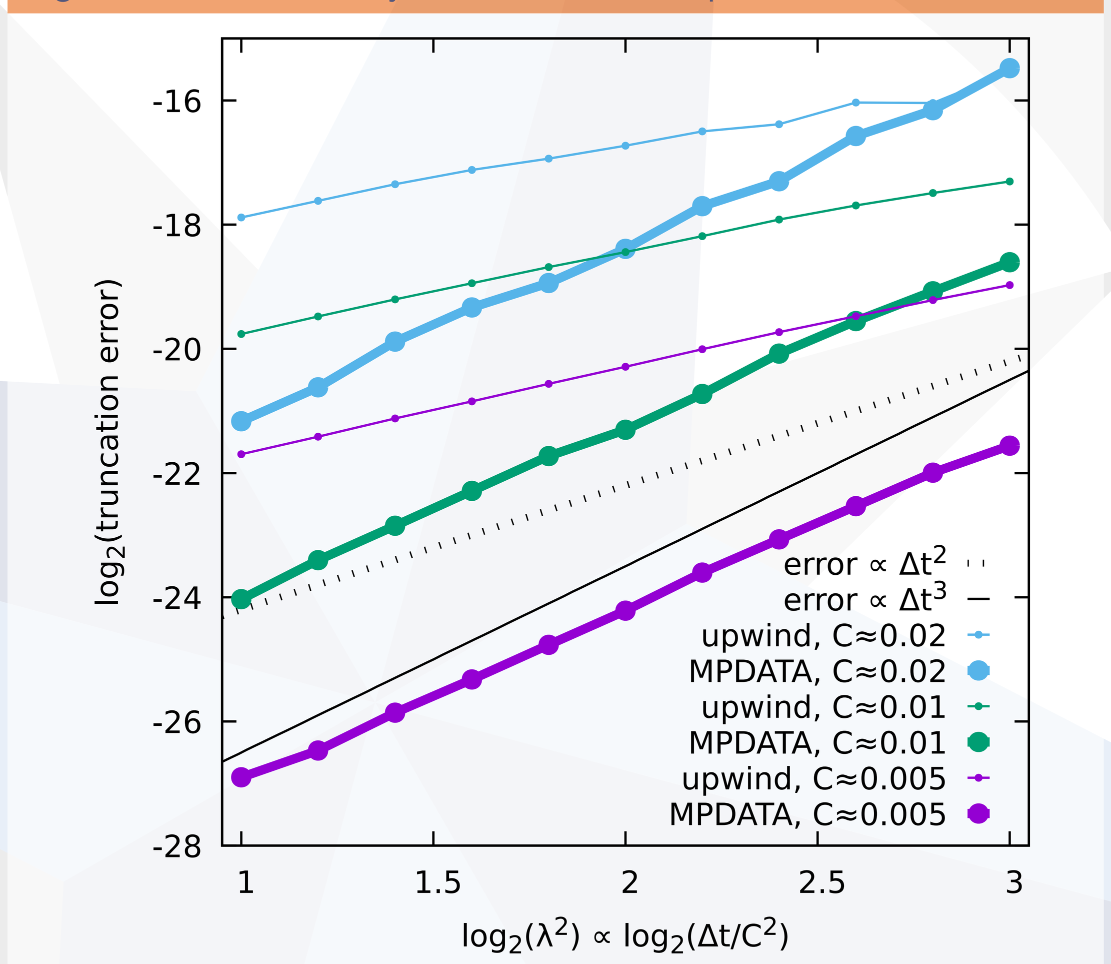
Comparison of a numerical solution obtained with MPDATA with the corresponding analytical solution (i.e., the Black-Scholes formula). Instrument parameters: a bought option with strike $K_1 = 0.75\%$ and a sold option with strike $K_2 = 1.75\%$, 6-month tenure (time to expiry), risk-free rate $r = 0.8\%$, volatility $\sigma = 0.6$.

Fig. 2: Solution accuracy in terms of the spatial discretisation



Truncation error as a function of the Courant number $C = u\frac{\Delta t}{\Delta x}$ which, for fixed λ^2, is proportional to the gridstep. Thin lines correspond to the basic upwind scheme (first iteration of MPDATA only), thick lines correspond to results obtained with one corrective iteration of MPDATA. Three datasets plotted for three different values of λ^2. The dotted and solid black lines depict the slopes corresponding to first-order and second-order convergence.

Fig. 3: Solution accuracy in terms of the temporal discretisation



Truncation error as a function of the λ^2 parameter which, for fixed C, is proportional to the timestep. Three datasets plotted for three different values of C (values given approximately as the solution procedure adjusts the requested value so that the number of timesteps is an integer). Other plot elements are as in Fig. 2.

References

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