MPDATA meets Black-Scholes: derivative pricing as a transport problem

Sylwester Arabas and Ahmad Farhat

background, affiliations

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(Models Dev. Team)

↙ AETHON, Athens
(H2020 MoveWise project)

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Jagiellonian Univ. Cracow
(from Oct. 2018)

↘ HSBC, Cracow
(Quant Team)

this talk (arXiv:1607.01751)

- MPDATA
- libmpdata++
- derivative pricing as a transport problem

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- MPDATA
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in line with the proposal put forward in Duffy 2004

to investigate robust and effective numerical schemes documented in the computational fluid dynamics literature as alternatives to commonly used numerical schemes in financial engineering, with the aim of “improving the finite difference methods gene pool as it were.”

(“A critique of the Crank-Nicolson scheme, strengths and weaknesses for financial instrument pricing”, WILMOTT 4)

MPDATA

MPDATA in a nutshell (Smolarkiewicz 1983, 1984, . . .)

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$$F(\psi_L, \psi_R, \mathcal{C}) = \max(\mathcal{C}, 0) \cdot \psi_L + \min(\mathcal{C}, 0) \cdot \psi_R$$
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MPDATA: reverse numerical diffusion by integrating the antidiffusive flux using upwind (in a corrective iteration)

$$\mathcal{C}'_{i+1/2} = (|\mathcal{C}_{i+1/2}| - \mathcal{C}^2_{i+1/2})A_{i+1/2}$$
$$A_{i+1/2} = \frac{\psi_{i+1} - \psi_i}{\psi_{i+1} + \psi_i}$$

MPDATA: key features (review: e.g. Smolarkiewicz 2006)

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- High-Order Accuracy:
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- Monotonicity:
  with Flux-Corrected Transport option

libmpdata++

libmpdata++: new open-source implementation in C++

Jaruga et al. 2015

Geosci. Model Dev., 8, 1005–1032, 2015
www.geosci-model-dev.net/8/1005/2015/
doi:10.5194/gmd-8-1005-2015

Geoscientific Model Development
Open Access

libmpdata++ 1.0: a library of parallel MPDATA solvers for systems of generalised transport equations

A. Jaruga[1], S. Arabas[1], D. Jarecka[1,2], H. Pawlowska[1], P. K. Smolarkiewicz[3], and M. Waruszewski[1]

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[2]National Center for Atmospheric Research, Boulder, CO, USA
[3]European Centre for Medium-Range Weather Forecasts, Reading, UK

libmpdata++: generalised transport equation

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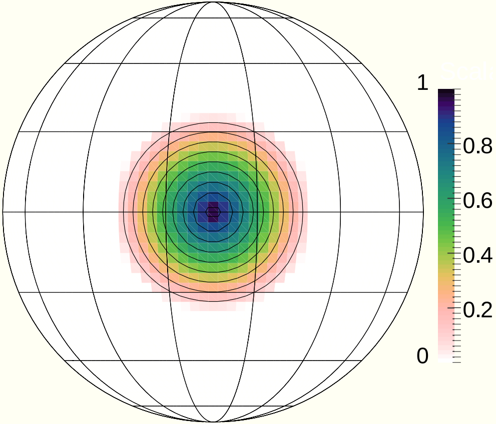

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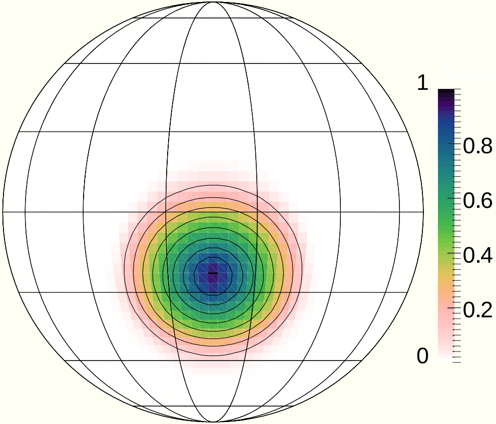

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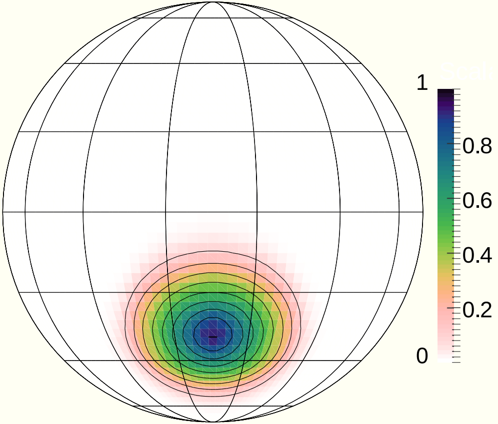

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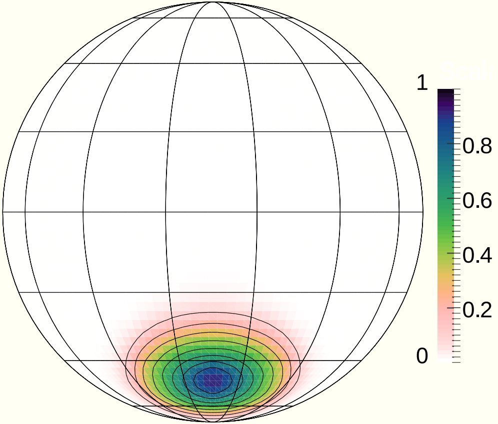

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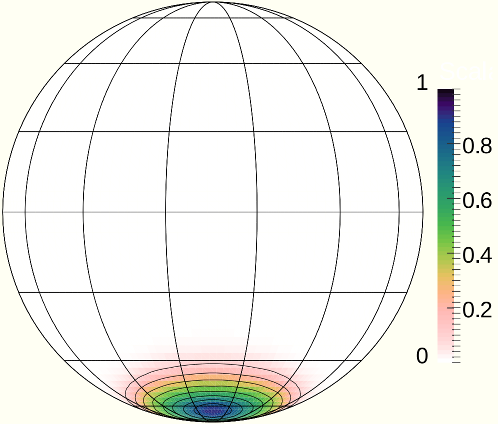

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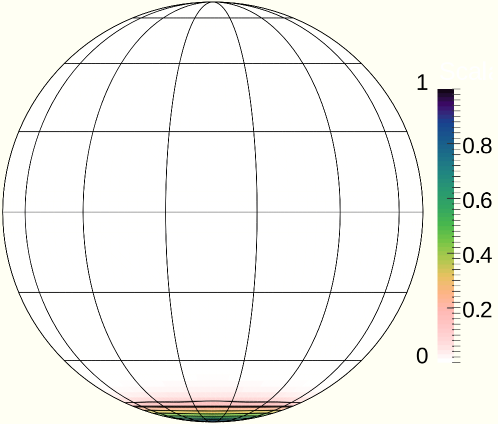

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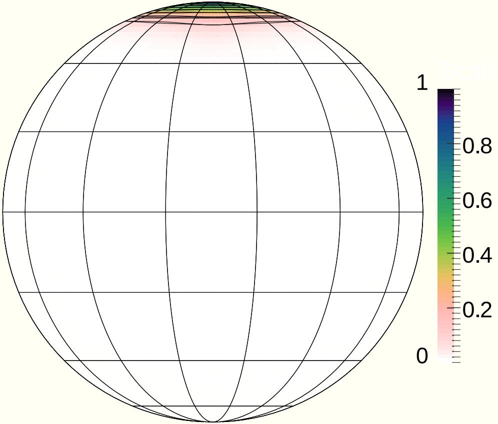

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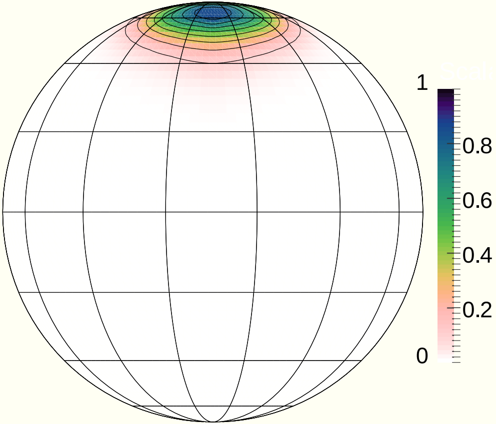

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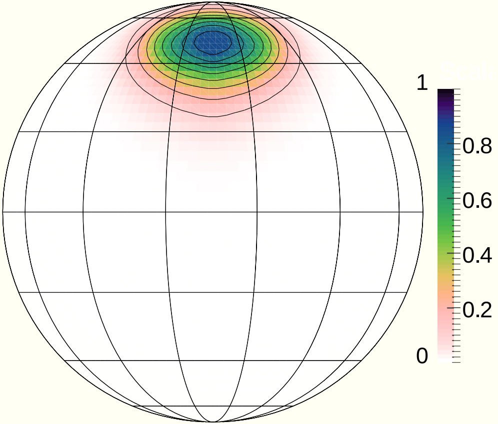

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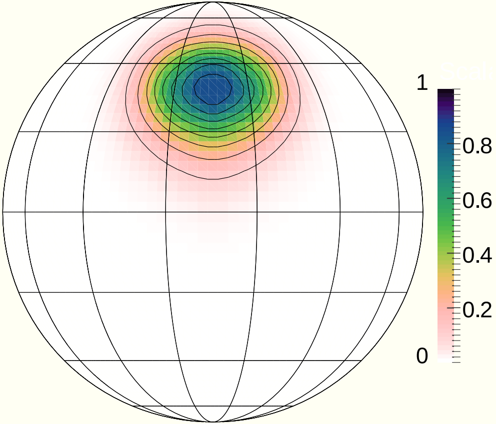

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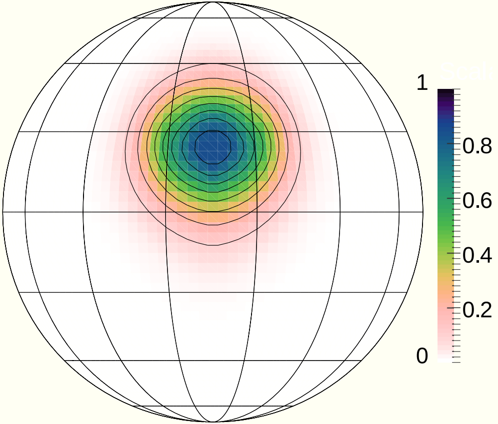

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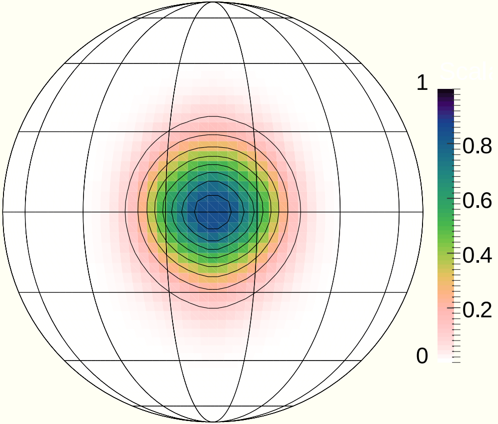

libmpdata++: summary & some technicalities

key features (as of v1.0):

- reusable – API documented in the paper; out-of-tree setups
- comprehensive set of MPDATA opts (incl. FCT, infinite-gauge, ...)
- 1D, 2D & 3D integration; optional coordinate transformation
- four types of solvers:
  - adv (homogeneous advection)
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derivative pricing as a transport problem

Black-Scholes equation and pricing formulæ

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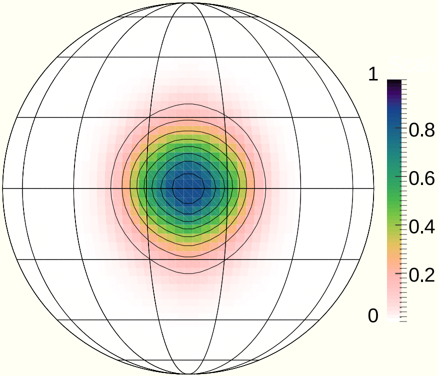
1
0.8
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0.4
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Black-Scholes ⇝ ("advection-only") transport problem

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re last step: Smolarkiewicz and Clark (1986, JCP), Sousa (2009, IJNMF), Smolarkiewicz and Szmelter (2005, JCP), Cristiani (2015, JCSMD)

same trick!

MPDATA in a nutshell (Smolarkiewicz 1983, 1984, . . .)

transport PDE: $\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(v\psi) = 0$

$$\psi_i^{n+1} = \psi_i^n - \left[F(\psi_i^n, \psi_{i+1}^n, \mathcal{C}_{i+1/2}) - F(\psi_{i-1}^n, \psi_i^n, \mathcal{C}_{i-1/2})\right]$$
$$F(\psi_L, \psi_R, \mathcal{C}) = \max(\mathcal{C}, 0) \cdot \psi_L + \min(\mathcal{C}, 0) \cdot \psi_R$$
$$\mathcal{C} = v\Delta t/\Delta x$$

← upwind

modified eq.: $\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(v\psi) + \underbrace{K\frac{\partial^2 \psi}{\partial x^2}}_{\text{numerical diffusion}} + \ldots = 0$ ← MEA

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(v\psi) + \frac{\partial}{\partial x}\underbrace{\left[\left(-\frac{K}{\psi}\frac{\partial \psi}{\partial x}\right)\psi\right]}_{\text{antidiffusive flux}} = 0$$

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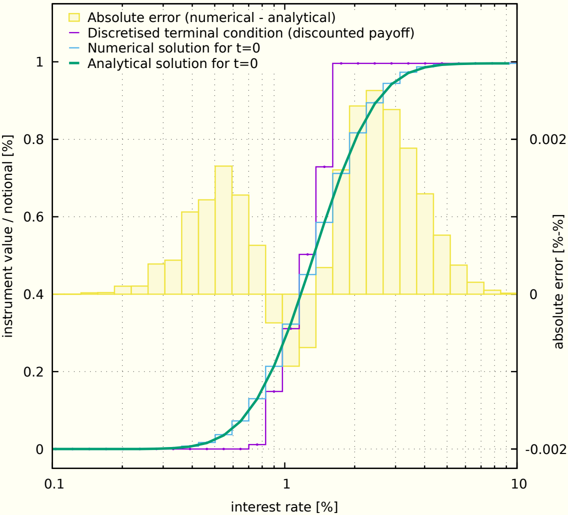
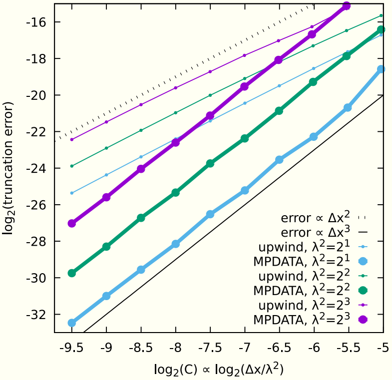
MPDATA meets Black-Scholes: test case

- payoff function: corridor
- truncation error est. (ψ_a: B-S formula):

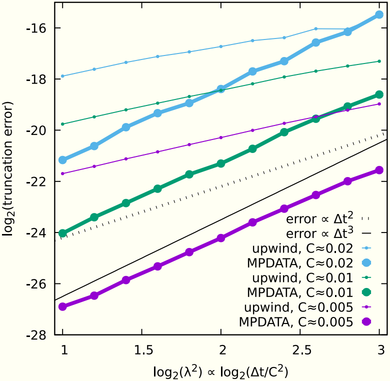
$$E = \sqrt{\sum_{i=1}^{n_x} \left[\psi_n(x_i) - \psi_a(x_i)\right]^2 / (n_x \cdot n_t)}\Bigg|_{t=0}$$



MPDATA meets Black-Scholes: convergence analysis



Truncation error as a function of the Courant number $C = u\frac{\Delta t}{\Delta x}$ which, for fixed λ^2, is proportional to the gridstep.



Truncation error as a function of the λ^2 parameter which, for fixed C, is proportional to the timestep.

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- Black-Scholes put formula = *"standard model for the transport of an unreactive solute in a soil column"* $\rightsquigarrow$ Hogarth et al. (1990, Comp. Math. Applic.)

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- thanks to organisers, thank you for and your attention!