PATH-DEPENDENT OPTION VALUATION WITH PDES (TEST CASE FOR MULTI-DIMENSIONAL ADVECTION SCHEMES?)

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BLACK-SCHOLES(-MERTON) PDE

🏆nobelprize.org/prizes/economic-sciences/1997/summary

THE NOBEL PRIZE

Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997

💹 BLACK-SCHOLES(-MERTON) PDE

- SDE for proportional asset returns: $dS_t/S_t = \mu dt + \sigma dW_t$
- option price process: $f(S_t, t)$
- delta-hedged portfolio (sell option, buy Δ_t assets): $\Pi_t = -f(S_t, t) + \Delta_t S_t$
- Itô's lemma applied to $f(S_t, t)$: SDE ⇝ PDE for df
- no arbitrage (risk-free interest rate r): $d\Pi_t/dt = \Pi_t r$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} - rf = 0$$

(terminal value problem, analytic solutions for vanilla options)

🌊 B-S-(M) PDE AS A TRANSPORT PROBLEM

📈 $$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} - rf = 0$$ 📉

$x = \ln S \longrightarrow$ *[Math Processing Error]*

$\psi = e^{-rt} f$ (discounted value) $\longrightarrow \frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} - \nu\frac{\partial^2 \psi}{\partial x^2} = 0$

diffusive pseudo-velocity $\longrightarrow$ *[Math Processing Error]*

B-S-(M) PDE WITH UPWIND DISCRETIZATION

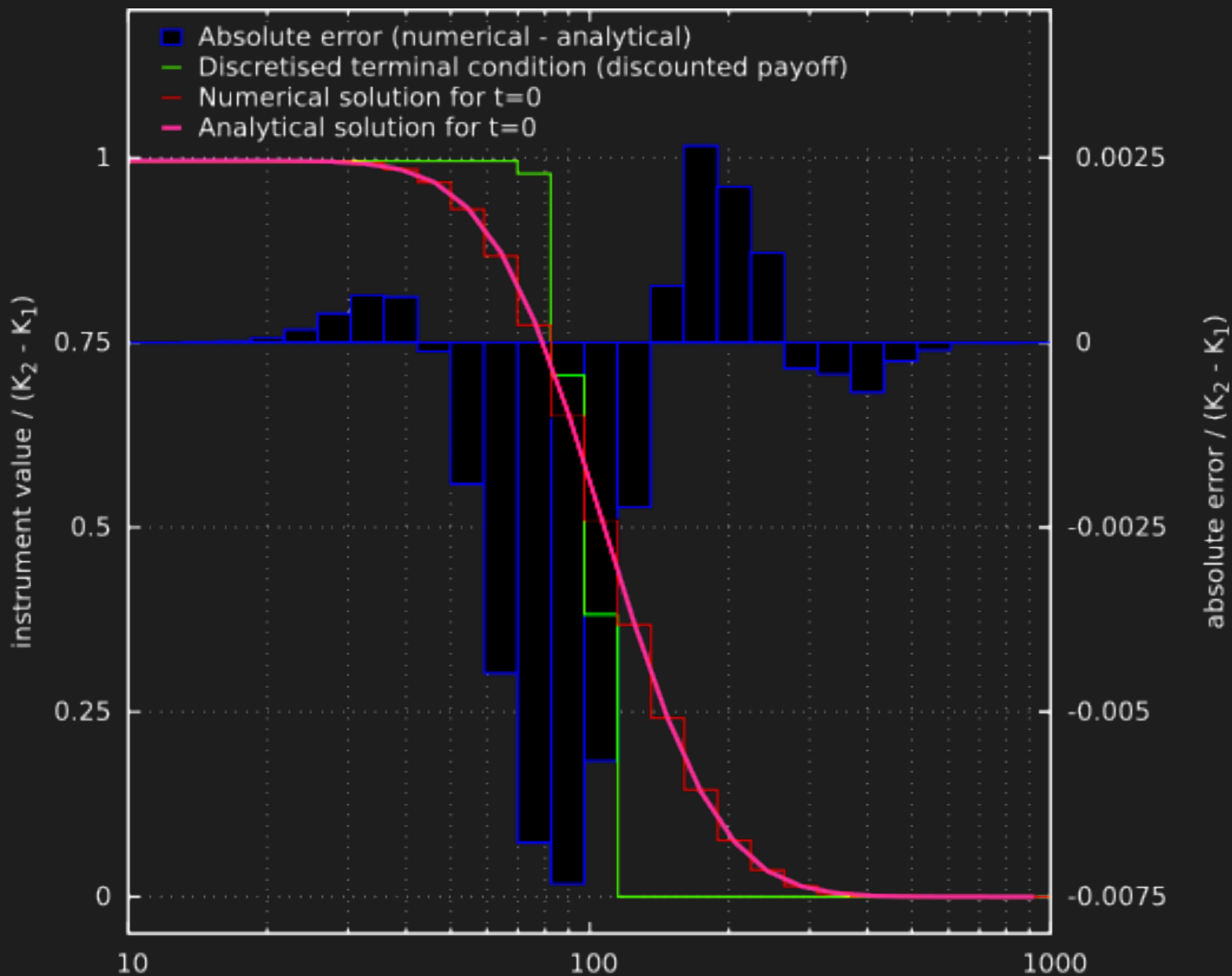
- stability condition (Courant no.) $\longrightarrow$ *[Math Processing Error]*

  since A is bounded for $\psi \geq 0$, the condition $\longrightarrow \frac{1}{\sigma^2}\frac{\Delta x^2}{\Delta t} > 2$

- instrument pricing procedure:
  - terminal condition for $\psi|_{t=T} \longrightarrow$ discounted payoff (put/call option[s])
  - integrate **backward in time**! ($T \longrightarrow 0$)
  - $\psi|_{t=0} \longrightarrow$ value of the option as a function of the asset

- non-oscillatory, positive-definite scheme $\longrightarrow$ monotonic price, optionality!

💹 EXAMPLE: CORRIDOR (BUY OPTION TO SELL FOR K_2, SELL OPTION TO SELL FOR K_1)



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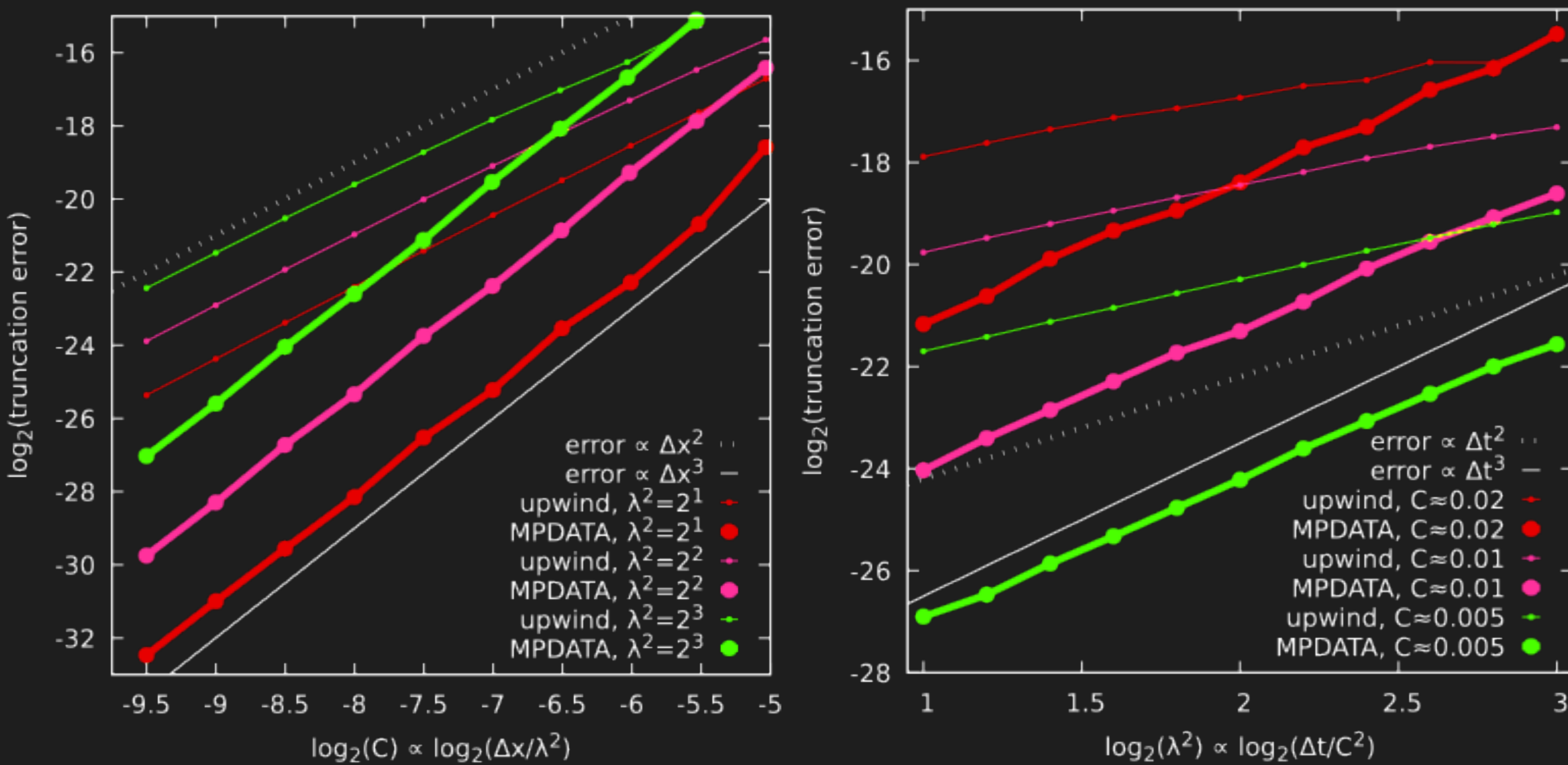
🏁 "SPATIAL" AND TEMPORAL CONVERGENCE FOR MPDATA/FCT (ITERATIVE UPWIND)[1]



1. S. Arabas and A. Farhat, "Derivative Pricing as a Transport Problem: MPDATA Solutions to Black–Scholes-Type Equations," J. Comp. Appl. Math. 373 (2020)

🗺️ "GEOGRAPHY" OF DERIVATIVE PRICING PROBLEMS

🌍 **European options**

⟶ can be exercised at expiration

⟶ Black-Scholes(-Merton) analytic solutions

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} - rf = 0$$

🌎 **American options**

⟶ can be exercised **any time** prior to expiration

⟶ Black-Sscholes(-Merton) **inequality** (free boundary problem)

[Math Processing Error]

🌏 **Asian options** (Eurasian here, also Amerasian/Hawaiian)

⟶ payoff depends on the **average asset value**

⟶ path-dependent problem, **2D B-S(-M)** (augmented state-space)

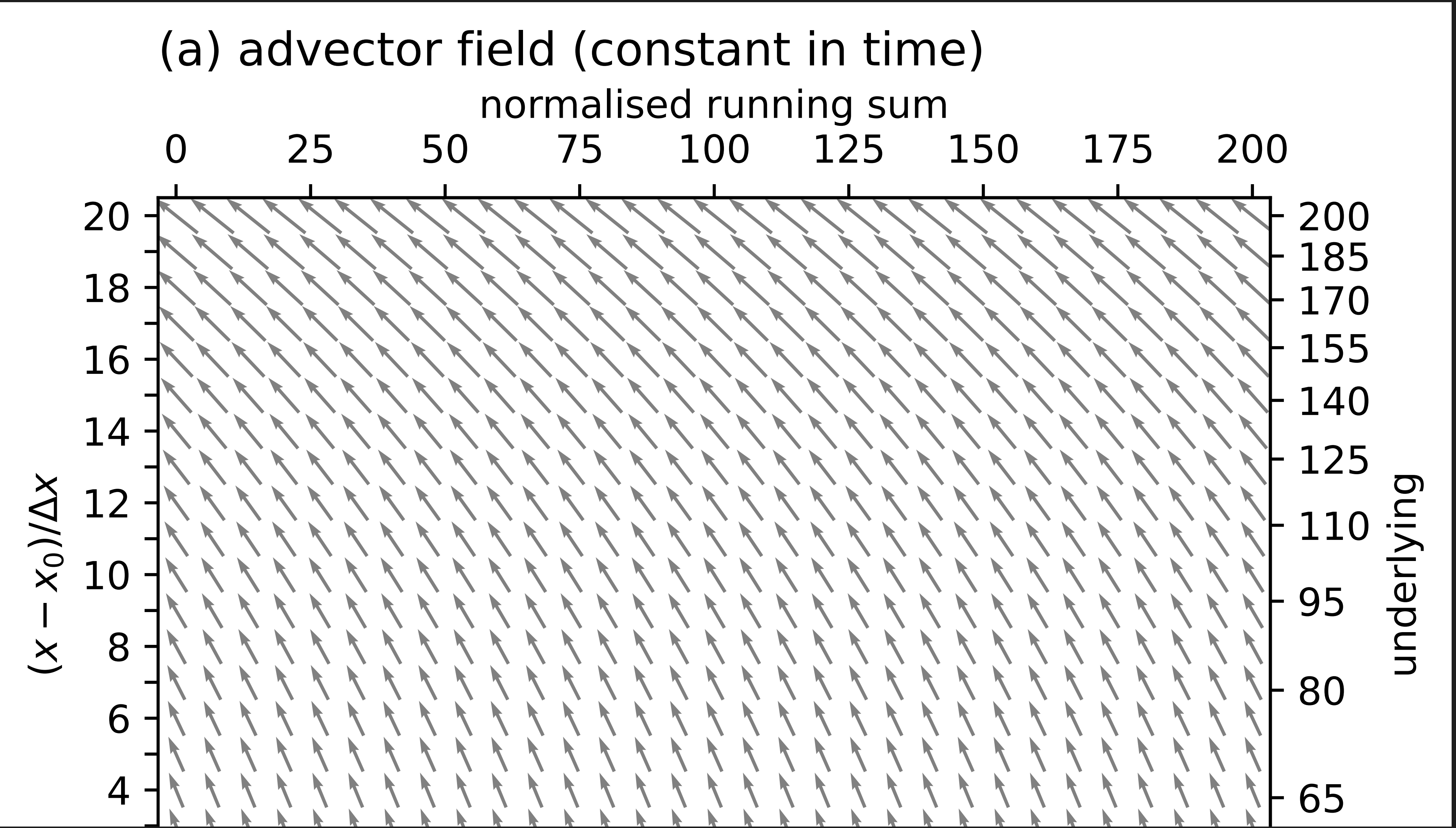
$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial A} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} - rf = 0 \qquad A = \frac{1}{T}\int_0^t S(\tau)d\tau \;\rightsquigarrow\; v = S/T$$

🌊 ASIAN OPTION PRICING AS A TRANSPORT PROBLEM

$$\partial_t \psi + \nabla \cdot (\vec{u}\psi) = 0$$

[Math Processing Error]

VALUATION EXAMPLE: OPTION TO BUY WITH F(S,A,T)=MAX(A(T)−K, 0) PAYOFF



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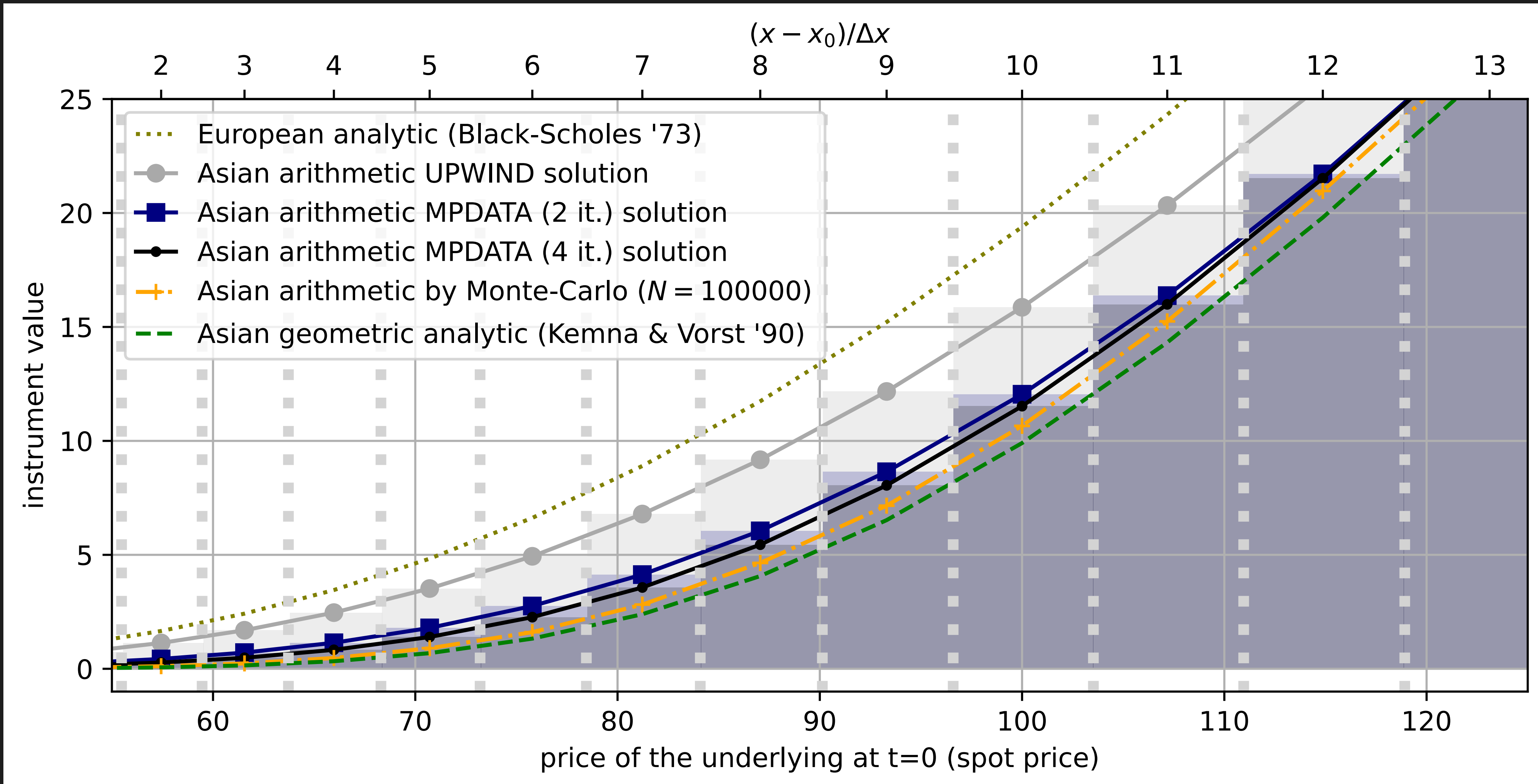

Monte-Carlo benchmark; analytic only for geometric mean

- 📌 quant-finance application of the non-oscillatory **MPDATA**[1]
- 📌 single-operator numerics for the B-S(-M) valuation framework (1D or 2D)
- 📌 positive-definite ⟶ optionality; non-oscillatory ⟶ monotone price
- 📌 open-source **pure-Python** implementation: **PyMPDATA** (⟶ Numba[-MPI])
- 📌 Jupyter/Colab notebooks (PyMPDATA @ Software Afternoon session!)

1. W. W. Grabowski and P. K. Smolarkiewicz, "Monotone Finite-Difference Approximations to the Advection-Condensation Problem," *Mon. Weather Rev.* 118 (1990)

🌊 CFD ⇝ 💹 QUANTITATIVE FINANCE

[proposal] "*to investigate robust and effective numerical schemes documented in the* ***computational fluid dynamics*** *literature as alternatives to commonly used numerical schemes in financial engineering, with the aim of* ***improving the finite difference methods gene pool****…*"[1]

💹 QUANTITATIVE FINANCE ⇝ 🌊 CFD

the 2D Asian option option problem as a test case for advection schemes (vs. **Monte-Carlo** & ≈**analytic**)?

1. D. Duffy, "A Critique of the Crank-Nicolson Scheme, Strengths and Weaknesses for Financial Instrument Pricing," *WILMOTT 4 (2004)*

THANK YOU FOR YOUR ATTENTION

- **European & American** option pricing with MPDATA: **Arabas & Farhat 2020**
🔗arXiv:1607.01751 (JCAM 2020)
- **Asian** option pricing with MPDATA: **Magnuszewski & Arabas 2025**
🔗arXiv:2505.24435

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